WAVELETS USAGE FOR IMAGE COMPRESSION

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Keywords: wavelet, image compression, fast wavelet transform

Articolul prezinta notiuni de baza privind teoria undinelor si utilizarea lor pentru realizarea comprimarii de imagini. Sunt prezentate functii C pentru determinarea valorilor functiei de scalare in puncte diadice si pentru determinarea coeficientilor transformatei undine a unui semnal discret. In final se testeaza eficienta compresiei unor date folosind diferite tipuri de undine.

The paper describes the basic notions about the wavelets and their usage in image compression applications. There are also presented C routines that compute the values of the scaling function in dyadic points and determine the wavelets coefficients associated to a discrete signal. These functions are finally used to evaluate the efficiency of data compression with various types of wavelets.

1. Theoretical background

The wavelets theory describes methods to decompose a general function *f* in terms of other simpler component functions, which could be interpreted as "building blocks" of *f*. The two essential properties of such a component function are the *space localization* and the *scale of representation*.

Let $L^2(R)$ be the space of square integrable functions i.e.

$$L^2(R) = \{ f : R \to C | \int_{-\infty}^{\infty} |f(x)|^2 dx < \infty \}$$

$L^2(R)$ is a Hilbert space relative to the scalar product:

$$< f, g >= \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx$$

Wavelet expansions have properties that differentiate them from other methods of signal decomposition. For example a function (signal) $f \in L^2(R)$ could be expanded using Fourier series:

$$f(x) = b_0 + \sum_{k \geq 1} (b_k \cos(2\pi kx) + a_k \sin(2\pi kx)) \quad (1)$$

This type of representation is not suitable to be used in case of functions with finite support (which are zero outside a finite interval). In this case an accurate Fourier expansion should have a lot of coefficients in order to cancel the effect of the sinusoids beyond the finite interval. In contrast with Fourier series (which analyze signals in frequency space) the wavelets are well localized in space; that is why they are adequate to describe signal properties in physical space. Many types of wavelets have finite support so, to accurately represent an arbitrary finite support function, a small number of coefficients is often sufficient.

Wavelets are used in various application fields as signal processing or mathematical physics (they are valuable tools for solving differential equations).

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A wavelet expansion is formed from copies with different translations (positions) and dilations (scales) of a fixed function $\psi \in L^2(R)$ called the "mother wavelet".

$$\psi_{a,b}(x) = \frac{1}{\sqrt{|a|}}\psi(\frac{x-b}{a}) \quad where \quad a,b \in R, \quad a \neq 0 \qquad (2)$$

a is the dilation parameter and b the translation parameter of the wavelet.

The continuous wavelet transform of a function $f \in L^2(R)$ is defined by:

$$W(a,b) = < f, \psi_{a,b} > \quad \forall a,b \in R, \quad a \neq 0$$

It has be proven that, in case of wavelets that satisfy the condition

$$C_\psi = \int_{-\infty}^{\infty} \frac{|\psi(\omega)|^2}{\omega} d\omega < \infty$$

the continuous wavelet transform can be inverted and the inverse transform is given by:

$$f(x) = \frac{1}{C_\psi} \int\int W(a,b)\psi_{a,b}(x) \frac{dadb}{a^2}$$

This way to introduce wavelets (the continuous wavelet transform) is the most general. Many application classes do not require the determination of W for all the possible values of a and b but only for discrete values. The most widely used is the set of dyadic values, where: $a=2^j$, $b/a=k$, $j, k \in Z$. The dyadic values have a binary representation with a finite number of digits. The transform that uses only dyadic values for a and b is called the discrete wavelet transform. It is characteristic to another way of interpreting wavelets called "multiresolution approximation". There are equally cases when a and b belong to other subsets of R (different from the dyadic values subset).

The *multiresolution approximation* is a formalism used to emphasize certain wavelet properties. The signal *f* that has to be expanded is represented by summing signals at different resolutions. The above mentioned technique consists of building a sequence of such spaces (subspaces of $L^2(R)$) each of them containing an approximation of *f*, at a certain resolution.

A multiresolution expression of a function $f(x) \in L^2(R)$ is constructed considering a sequence of approximation spaces of f(x):

$$\ldots \quad V_{-2} \subseteq V_{-1} \subseteq V_0 \subseteq V_1 \subseteq V_2 \subseteq \quad \ldots$$

having the following properties:

P0. $\exists\ \varphi(x) \in V_0$ so that the functions from $\{\varphi(x-k) | k \in Z\}$ form an orthonormal basis of V_0. $\varphi(x)$ is called the *scaling function*.

P1. $V_i \subseteq V_{i+1}, \forall i \in Z$

P2. $\cap_{i \in Z} V_i = \{0\}$

P3. $\cup_{i \in Z} V_i$ is dense in $L^2(R)$

P4. $f(x) \in V_i \Rightarrow f(x-k) \in V_i, \forall k \in Z$

P5. $f(x) \in V_i$ if and only if $f(2x) \in V_{i+1}$

P6. $\exists\ \varphi(x) \in V_0$ such that $\forall j \in Z \Rightarrow \{\varphi_{j,k}(x) = 2^{j/2}\varphi(2^j x - k) \mid j,k \in Z\}$ forms an orthonormal basis for V_j (i.e. $<\varphi_{j,p}, \varphi_{j,q}> = \int \varphi_{j,p}(x)\varphi_{j,q}(x)dx = \delta_{p,q}, \forall p,q \in Z$)

From the fact that $\varphi(x) \in V_0 \subseteq V_1$ it results that $\varphi(x)$ could be written as a linear combination of $\varphi(2x-k)$, $k \in Z$

$$\varphi(x) = \sum_{k \in Z} \varphi(2x-k) \qquad (3)$$

The formula above called “the dilation equation” specifies a fundamental selfsimilarity property of the scaling function and determines completely (modulo a multiplicative factor) the value of φ(x) in all dyadic points.

Condition (P6) written for j=0 gives:

$$\int \varphi(x-k)\overline{\varphi(x)}dx = \delta_{k,0}, \qquad \forall k \in Z$$

where:

$$\delta_{p,q} = \begin{cases} 1 & if \quad p = q \\ 0 & if \quad p \neq q \end{cases}$$

is the Kronecker symbol.

Substituting φ(x) and φ(x-k) with values given by (3) results:

$$\int \sum_{p\in Z} C_p \varphi(2x-2k-p)\overline{\sum_{q\in Z} C_q \varphi(2x-q)}dx = \delta_{k,0}$$

but:

$$\sum_{p\in Z}\sum_{q\in Z} C_p\overline{C_q}\int \varphi(2x-2k-p)\overline{\varphi(2x-q)}dx = \sum_{p\in Z}\sum_{q\in Z} C_p\overline{C_q}2^{-1}\int \varphi(x-2k-p)\varphi(x-q)dx$$

therefore:

$$2^{-1}\sum_{p\in Z}\sum_{q\in Z} C_p\overline{C_q}\delta_{2k+p,q} = \delta_{k,0}$$

that is equivalent with:

$$\sum_{i\in Z} C_i\overline{C_{2k+i}} = 2\delta_{k,0} \tag{4}$$

Another condition that must be accomplished by the coefficients C_i, $i\in Z$, derives from the assumption that: $\int\varphi(x)dx=1$. Integrating the dilation equation:

$$\int \varphi(x)dx = \sum_{k\in Z} C_k \int \varphi(2x-k)dx = 2^{-1}\sum_{k\in Z} C_k \int \varphi(x)dx$$

i.e.

$$\sum_{k\in Z} C_k = 2 \tag{5}$$

Other condition that is usually imposed to φ(x) (called the “partition of unity condition”) is:

$$\sum_{k\in Z} \varphi(x-k) = 1, \qquad \forall x \in R$$

It leads to a condition for the coefficients C_k, $k\in Z$

$$\sum_{k\in Z} (-1)^k C_k = 0 \tag{6}$$

It is not possible to combine only the functions from the orthogonal bases of V_j, $j\in Z$, in order to obtain an arbitrary function in $L^2(R)$. The construction of a basis in $L^2(R)$ is done as follows.

The space V_0 has already a basis formed from the functions $\{\varphi(x-k)|k\in Z\}$, to complete the orthonormal basis of V_1, another function space W_0 (the orthogonal complement of V_1 relative to V_0) will be considered.

$V_1=V_0\oplus W_0$, where $\oplus$ denotes the direct sum of Hilbert spaces. Each element from V_1 could be uniquely written as a sum from an element of V_0 and an element of W_0. The basis of W_0 is a set of functions denoted: $\{\psi(x-k)|k\in Z\}$.

$$\psi(x) \in V_1 \Rightarrow \psi(x) = \sum_k h_k \varphi(2x-k) \tag{7}$$

Therefore V_1 contains functions f(x) that are combinations of functions φ(x) and ψ(x). The previously mentioned process could continue: $V_2=V_1 \oplus W_1$, …which leads to:

$$L^2(R) = V_0 \oplus (\overset{\infty}{\underset{i=0}{\oplus}} W_i)$$

In conclusion every function $f(x) \in L^2(R)$ could be written as a linear combination of functions from $\{\varphi(x-k) | k \in Z\}$ and $\{2^{j/2}\psi(2^j x-k) | j, k \in Z, j \geq 0\}$. The spaces W_j, contain the "detail" information necessary to obtain an approximation at resolution j+1 from an approximation at resolution j of the signal *f*.

Further, $V_0=V_{-1} \oplus W_{-1}$, $V_{-1}=V_{-2} \oplus W_{-2}$, … so that:

$$L^2(R) = \overset{\infty}{\underset{-\infty}{\oplus}} W_i \tag{8}$$

that is the functions $\{\psi_{j,k}(x)= 2^{j/2}\psi(2^j x-k) | j, k \in Z\}$ constitute a basis in $L^2(R)$. Any function $f(x) \in L^2(R)$ could be expanded as:

$$f(x) = \sum_{j,k \in Z} \lambda_{j,k} \psi_{j,k}(x) \tag{9}$$

The function $\psi(x)= \psi_{0,0}(x)$ with the property that $\{\psi(x-k) | k \in Z\}$ is a basis of W_0 is called the "mother wavelet" and the numbers $\lambda_{j,k}$ are the wavelet coefficients of f(x). Equation (9) specifies the discrete wavelet transform of the signal *f*.

It is not mandatory that $W_i \perp V_i$ but the wavelet coefficients are most easily computed in case of orthogonal wavelets. In this later case, the mother wavelet satisfies the conditions:

$$\int \psi(x)\psi(x-l)dx = \delta_{l,0} \qquad \forall l \in Z$$

and the wavelet coefficients are given by:

$$\lambda_{j,k} = < f, \psi_{j,k} > = \int f(x)\psi_{j,k}(x)dx$$

It has been proved that a sufficient condition for ψ to be an orthogonal wavelet is:

$$h_k = (-1)^k \overline{C_{1-k}} \qquad k \in Z \tag{10}$$

2. Computing the scaling function values in dyadic points

An important property of wavelets is the precision they are able to approximate a given function. For example, the zero-order wavelets can match only piecewise constant functions, and the first-order wavelets match the piecewise linear functions. The order of a wavelet measures his ability to match a polynomial. A wavelet has order p if and only if his first p moments are zero:

$$\int x^m \psi(x)dx = 0 \qquad \forall 0 \leq m < p \tag{11}$$

An important class of wavelets is formed by those that have compact support. Daubechies ([2]) and Chui ([1]) constructed such categories of wavelets. The specification of such wavelets requires determination of coefficients C_i from the dilation equation (3). An example are the original Daubechies wavelets. They have order p and a support interval of length N=2p+1. In the case of Daubechies finite support wavelets, the conditions (11) produce the following conditions relative to the coefficients C_i:

$$\sum_k (-1)^k k^m C_k = 0, \qquad 0 \leq m < p \tag{12}$$

When p=1 the support interval is [0,3] and only four coefficients C_0, C_1, C_2, C_3, have non-null values. The restrictions already mentioned in (4), (5) and (6) give for C_i, $0 \le i \le N$ the values: $C_0=(1+\sqrt{3})/4$; $C_1=(3+\sqrt{3})/4$; $C_2=(3-\sqrt{3})/4$; $C_3=(1-\sqrt{3})/4$. This wavelet with four non-null coefficients is called the D_4 wavelet.

In order to compute the values of the scaling function associated to D_4 in the dyadic points from interval [0,1] ($x=k2^{-j}$, $j \in Z$, $0 \le k < 2^j$) we start from the equation (3). The values of φ in integer points must initially be known. It will be supposed that: $\varphi(x)=0$ for $x \in R-(0,4)$. In order to determine $\varphi(1)$ and $\varphi(2)$ the values x=1 and x=2 shall be substituted in the dilation equation. It results:

$$C_1\varphi(1)+C_0\varphi(2)=\varphi(1)$$
$$C_3\varphi(1)+C_2\varphi(2)=\varphi(2)$$

This means that $[\varphi(1), \varphi(2)]^T$ is an eigenvector of the matrix $\begin{bmatrix} C_1 & C_0 \\ C_3 & C_2 \end{bmatrix}$ and its associated eigenvalue is 1. Indeed:

$$\det\left(\begin{bmatrix} C_1 & C_0 \\ C_3 & C_2 \end{bmatrix} - I\right) = 0 \quad \Leftrightarrow \quad (C_1-1)(C_2-1) - C_0C_3 = 0$$

and the second equation could easily be demonstrated by adding and subtracting (5) and (6).
So, the eigenspace associated to the eigenvalue 1 is formed from the vectors $[\xi_1, \xi_2]^T$ where $(C_1-1)\xi_1+C_0\xi_2=0$. If $\xi_1=\alpha C_0$ then $\xi_2=\alpha(1-C_1)/C_0=\alpha C_3$ (because by adding (5) and (6) results $C_1+C_3=1$).
That is $\varphi(1)$ and $\varphi(2)$ are determined modulo a multiplicative factor α: $\varphi(1)=\alpha C_0$, $\varphi(2)=\alpha C_3$. The function from Figure 1 has $\alpha=2$.

Let consider the case of D_4 wavelet. Writing N times the dilation equation and retaining only the N+1 terms corresponding to the non-null values of the coefficients C_k results:

$$\begin{cases} \varphi(x) = C_0\varphi(2x) + C_1\varphi(2x-1) + C_2\varphi(2x-2) + C_3\varphi(2x-3) \\ \varphi(x+1) = C_0\varphi(2x+2) + C_1\varphi(2x+1) + C_2\varphi(2x) + C_3\varphi(2x-1) \\ \varphi(x+2) = C_0\varphi(2x+4) + C_1\varphi(2x+3) + C_2\varphi(2x+2) + C_3\varphi(2x+1) \end{cases} \quad (13)$$

Considering $x \in [0,1]$, the cases $x \in [0,1/2)$ and $x \in [1/2,1]$ will be separately treated.
If $x \in [0,1/2)$ equations (13) become:

$$\begin{cases} \varphi(x) = C_0\varphi(2x) \\ \varphi(x+1) = C_0\varphi(2x+2) + C_1\varphi(2x+1) + C_2\varphi(2x) \\ \varphi(x+2) = C_2\varphi(2x+2) + C_3\varphi(2x+1) \end{cases} \quad (14)$$

Similarly, for $x \in [1/2,1]$ equations (13) become:

$$\begin{cases} \varphi(x) = C_0\varphi(2x) + C_1\varphi(2x-1) \\ \varphi(x+1) = C_1\varphi(2x+2) + C_2\varphi(2x+1) + C_3\varphi(2x-1) \\ \varphi(x+2) = C_3\varphi(2x+1) \end{cases} \quad (15)$$

As a conclusion, the digits of the binary representation of x assist the decision of choosing between the sets of equations (14) or (15) that will be used to find $[\varphi(x),\varphi(x+1),\varphi(x+2)]$ at the next iteration of the algorithm.

The C implementation of a routine, which computes the values of the scaling function in a dyadic point, is further specified. The wavelet's support is I=[0,N]. The coefficients C_0, C_1, …, C_N are specified in a global array. KMAX is the maximum allowable number of digits for the binary representation of x. The routine "simultaneously" determines the values $\varphi(x+i)$, for $0 \le i < N$; these values are stored in the array v_fi.

```
double fi(double x)
{ int ind, k, j, f;
  char a[KMAX];
  if(x<=0.0 || x>=(double)N) return 0.0;
  /* initialization of fi(x) for x=0, 1, …, N */
  init_vfi();
  for(k=N-1, ind=0; k>0; k--)
    if(x>=(double)k) {
      ind=k;
      x -= (double)k;
      break;
    }
  for(k=0; k<KMAX && x!=0.0; k++)
    if(x<0.5) {a[k]=0; x=2.0*x;} else {a[k]=1; x=2.0*x-1;}
  for(j=k-1, f=1; j>=0; f=1-f, j--)
    if(a[j]==0) { /* case 0 <= x < 0.5 */
      v_fi[f][0]=c[0]*v_fi[1-f][0];
      v_fi[f][1]=c[0]*v_fi[1-f][2]+c[1]*v_fi[1-f][1]+c[2]*v_fi[1-f][0];
      v_fi[f][2]=c[2]*v_fi[1-f][2]+c[3]*v_fi[1-f][1];
    }
    else { /* case 0.5 <= x <= 1.0 */
      v_fi[f][0]=c[0]*v_fi[1-f][1]+c[1]*v_fi[1-f][0];
      v_fi[f][1]=c[1]*v_fi[1-f][2]+c[2]*v_fi[1-f][1]+c[3]*v_fi[1-f][0];
      v_fi[f][2]=c[3]*v_fi[1-f][2];
    }
  return v_fi[1-f][ind];
}
```

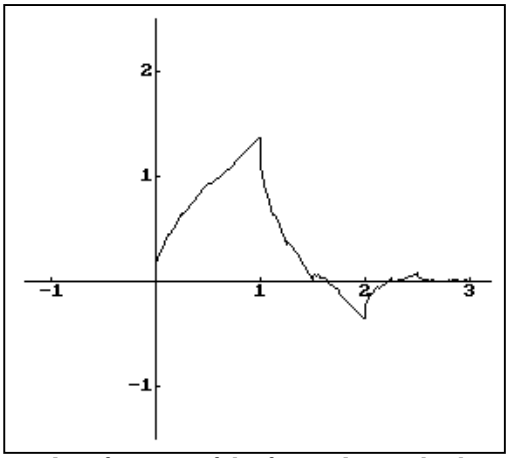

Figure 1 – The scaling function of the first order Daubechies wavelet

The values of the mother wavelet function could be computed using (10).

3. Determination of the wavelet coefficients associated to a given function

In the previous paragraphs it could be seen that a function $f(x) \in L^2(R)$ can be approximated by a function $v_j(x) \in V_j$ where:

$$v_j(x) = \sum_k \lambda_{j,k} \varphi_{j,k}(x), where \qquad \varphi_{j,k}(x) = 2^{j/2} \varphi(2^j x - k)$$

Due to the fact that $V_j = V_{j-1} \oplus W_{j-1}$, $\forall j \in Z$, it is equally true that:

$$v_j(x) = \sum_k \lambda_{j-1,k} \varphi_{j-1,k}(x) + \sum_k \mu_{j-1,k} \psi_{j-1,k}(x)$$

where:

$$\lambda_{j,k} = \sum_i C_{k-2i} \lambda_{j-1,i} + \sum_i h_{k-2i} \mu_{j-1,i}$$

When this last relation is repeatedly applied, it defines the inverse fast wavelet transform algorithm.

In the implementation below:

cd[k] stores the non-null coefficients C_k of the dilation equation

hd[k] stores the non-null coefficients of the mother wavelet (see (10))

lcd is the number of non-null coefficients of the dilation equation (4 in case of D_4 wavelet)

lhd is the number of non-null coefficients of the mother wavelet

mcd is the index of the first non-null coefficient of the dilation equation

mhd is the index of the first non-null coefficient of the mother wavelet

In order to satisfy condition $\Sigma(C_i)^2=2$, (see (4)) the wavelet coefficients where scaled with $1/\sqrt{2}$.

The algorithm starts with 2^n data values (discrete values of a function) stored in vector v. The corresponding wavelet coefficients are stored, by the direct wavelet transform algorithm, in vector w:

$w=[\lambda_{0,0}, \mu_{0,0}, \mu_{1,0}, \mu_{1,1}, \ldots, \mu_{k,0}, \ldots, \mu_{k,2^k-1}, \ldots \mu_{n-1,2^n-1}]$

```
void dfwt(int n)
/* direct fast wavelet transform
   v = input data vector
   w = output data vector
*/
{
  int j, k;
  long dlaj, l;
  long ind, ls;
  for(k=0, dlaj=1; k<n; k++) dlaj <<= 1;
  for(j=n-1; j>=0; dlaj>>=1, j--) {
  for(k=0; k<dlaj; k++) w[k]=0.0;
      for(l=0; l < (dlaj>>1); l++) {
        ind=modulo(2*l+mcd, j+1);
        for(k=0; k<lcd; k++) {
            w[l]=w[l]+cd[k]*v[ind];
            ind = modulo(ind+1, j+1);
        }
        ind=modulo(2*l + mhd, j+1);
        ls=l+(dlaj>>1);
        for(k=0; k<lhd; k++) {
            w[ls]=w[ls]+hd[k]*v[ind];
            ind = modulo(ind+1, j+1);
```

```
      }
    }
      for(k=0; k<(dlaj>>1); k++) v[k]=w[k];
  }
}

void ifwt(int n)
/* inverse fast wavelet transform
   w = input data vector
   v = output data vector
*/
{ int j, k, l, lb;
  long dlaj, ind;
  for(j=1, dlaj=2; j<=n; dlaj <<= 1, j++) {
      for(k=0; k<dlaj; k++) v[k]=0.0;
      for(k=0; k<dlaj; k++) {
        ind=modulo((long)((k-mcp)/2), j-1);
        lb=abs(k-mcp)%2;
        for(l=lb; l<lcp; l+=2) {
          v[k]=v[k]+cp[l]*w[ind];
          ind=modulo(ind-1, j-1);
        }
        ind=modulo((long)((k-mhp)/2), j-1);
        lb=abs(k-mhp)%2;
        for(l=lb; l<lhp; l+=2) {
          v[k]=v[k]+hp[l]*w[ind+(dlaj>>1)];
          ind=modulo(ind-1, j-1);
        }
      }
    for(k=0; k<dlaj; k++) w[k]=v[k];
  }
}
```

where the function modulo(k, j) computes k mod 2^j:

```
long modulo(long k, int j)
/* computes the remainder of k/2**j */
{
  long s; int i;
  for(i=0, s=0; i<j; i++) s = (s<<1) | 1;
  return k&s;
}
```

4. Evaluation of the compression method

In order to evaluate the efficiency of the wavelet based compression algorithm; it was taken into consideration the case of D_4 wavelet. There were considered packets of 64 “data values” in the range [0, 255]. These values could correspond to pixel intensity codes of grayscale images (with 256 intensity levels). Such data sets are similar to data blocks used for raster data codification by the sequential JPEG standard. (The standard uses a DCT compression method).

Five sets of values differently correlated (constant, linear, quadratic, cubic and randomly distributed) were considered. The data block was processed using the fast wavelet

transform and then reconstructed (using the inverse fast wavelet transform) starting from a number of NAPPROX coefficients having the largest absolute values.
The reconstruction error is computed by:

$$ERR\_TOTAL = \sum_{k=0}^{63} \left| B_k - B_k^{'} \right| \tag{16}$$

where B_k is the k-th data value of the original data block and $B_k^{'}$ is the corresponding value of the reconstructed block.

It has been proved ([9]) that choosing more and more approximating wavelet coefficients in the above mentioned order will produce a decreasing sequence of the total reconstruction errors.
Other performance measure could be the maximal percentage of deviation of a reconstructed data value by rapport to the corresponding original data value.

$$ER\_MAX = \max_k \left| \frac{B_k - B_k^{'}}{B_k} \right| * 100$$

Table 1 shows the total error, the total error per data sample and the maximal percentage of deviation for different values of NAPPROX.

Table 1

Performance indices of D_4 wavelet image compression

NAPPROX	ER_TOTAL	ER_TOTAL/POINT	ER_MAX	
1	0	0	0	constant
1	1024	16	3050	linear
8	65.6	1.02	108.3	
9	25.88	0.4	16.01	
11	0	0	0	
1	4125.51	64.46	1864.87	quadratic
7	279.53	4.37	1864.87	
18	15.87	0.25	102.45	
30	4.96	0.08	27.45	
63	0.13	0	0.04	
64	0	0	0	
1	943	14.74	793700	cubic
8	54.09	0.85	7301.79	
21	3.21	0.05	103.98	
52	0.2	0	46.2	
63	0	0	0	
1	4131	64.56	3019.92	random
20	2069	32.34	2153.62	
30	1296	20.25	211.58	
40	690.5	10.78	205.59	
50	357.79	5.59	79-28	
60	14.64	0.23	6.31	
63	0.54	0.01	0.27	
64	0	0	0	

Conclusions

The wavelet based image compression could be a valuable tool not only in data transmission but also in other application fields as: data bases for graphics processing, computer graphics (image synthesis, volumic data processing) or scientific visualization. It should be interesting to determine the wavelets that are most suitable to be used for different categories of images (noisy images, images with high frequencies etc.)

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